The Proportional Ordinal Shapley Solution for Pure Exchange Economies[*]

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Abstract

We define the *proportional ordinal Shapley* (the *POSh*) solution, an ordinal concept for pure exchange economies in the spirit of the Shapley value. Our construction is inspired by Hart and Mas-Colell's (1989) characterization of the Shapley value with the aid of a potential function. The *POSh* exists and is unique and essentially single-valued for a fairly general class of economies. It satisfies individual rationality, anonymity, and properties similar to the null-player and null-player out properties in transferable utility games. The *POSh* is immune to agents' manipulation of their initial endowments: It is not D-manipulable and does not suffer from the transfer paradox. Moreover, we characterize the *POSh* through a Harsanyi's (1959) system of dividends and, when agents' preferences are homothetic, through a weighted balanced contributions property à la Myerson (1980).

[*] We thank Mikel Alvarez-Mozos, Inés Macho-Stadler, Jordi Massò, David Wettstein, and participants at the 46th Simposio of Análisis Económico for their helpful comments. We gratefully acknowledge financial support from Ministerio de Economía y Competitividad and Feder (through the Severo Ochoa Programme for Centres of Excellence in R&D, CEX2019-000915-S, and the project PGC2018-094348-B-I00), Generalitat de Catalunya (2017SGR-711 and FI fellowship), and ICREA under the ICREA Academia programme. Declarations of interest: none

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Keywords: Shapley Value; Exchange economy; Ordinal solution; Potential

JEL Classification: D63; D50; C72

1 Introduction

Economists have long been proposing allocation rules for economic environments and evaluating them by different desiderata. Though no rule is advantageous under every criterion, some allocation rules arise as dominant solution concepts for specific economic environments, such as the Walrasian allocation rule for pure exchange economies and the Shapley value (Shapley, 1953) for coalitional games with transferable utility (TU). A natural question is whether we can extend solution concepts initially designed for a specific economic environment to another.

In this paper, we propose a solution concept for pure exchange economies in the spirit of the Shapley value, which satisfies many appealing properties and is characterized by several methods in the class of TU games. Our construction is inspired by Hart and Mas-Colell's (1989) characterization of the Shapley value with the aid of a potential function. This function assigns a number to every TU game with the only condition that the marginal contributions to the potential of all players add up to the worth of the grand coalition. Hart and Mas-Colell (1989) establish that there is only one such potential function and the vector of marginal contributions coincides with the Shapley value.

We follow a similar approach and associate a number to each pure exchange economy, the *potential* of this economy. Due to the absence of a numeraire commodity in these environments, we choose each agent's initial endowment as a yardstick to measure the variation of his welfare in a solution; this variation will be proportional to his initial endowment. The only condition that we impose on the potential function is the existence of an efficient allocation profile in the economy that satisfies that any agent is indifferent between that allocation and his marginal contribution to the potential times his initial endowment. That is, we require that it be possible for each agent to obtain their marginal contribution to the potential through an efficient allocation.

The construction of the potential of a pure exchange economy entails the simultaneous definition of the efficient allocation profiles that are equivalent for all the agents to their marginal contributions. These allocations are our solution for the economy. We name the set of these allocations the *proportional ordinal Shapley* (the *POSh*) solution. We include the word "ordinal" in the name of the solution because its first important characteristic is that, by construction, the *POSh* is an *ordinal solution*, that is, it is invariant to order-preserving transformations of the agents' utilities. Moreover, we show that the *POSh* solution is *unique* and *essentially single-valued*[1] in the set of exchange economies where the agents' preferences are reflexive, complete, transitive, strongly monotone, and continuous. It is also *individually rational.*

The *POSh* inherits several of the appealing properties of the Shapley value. In particular, it is *anonymous* with respect to the name of the agents (and it is also *neutral* with respect to the name of the commodities). Additionally, the *POSh* prescribes a zero bundle to any agent with zero endowments (these are "empty-bundle agents," we call them "empty agents" for short); that is, it satisfies the *empty-agent property.* Further, it satisfies the *empty-agent out property*, which requires that the presence of an empty agent does not influence the prescribed bundles for the rest of the agents. These properties are reminiscent of the null player property and the null player out property of the Shapley value (Derks and Haller, 1999).

To further highlight the links between the *POSh* and the Shapley value, we provide a characterization of the *POSh* using a system of *dividends* that is similar to the characterization of the Shapley value in TU games in terms of the Harsanyi's (1959) coalitional dividends. Also, when agents' preferences are homothetic, we characterize the *POSh* by efficiency and a weighted *balanced contributions property*, in the spirit of Myerson's (1980) characterization of the Shapley value. Finally, we relate the *POSh* and the "proportional Shapley value" for TU games (Besner, 2016,

[1] That is, if the *POSh* solution prescribes several allocations to an economy, every agent is indifferent among all these allocations. Moreover, any allocation that is indifferent for every agent to an allocation in the *POSh* solution is also in this set.

and Béal et al., 2018).

The *POSh* is immune to certain peculiarities suffered by several allocation rules for pure exchange economies, such as the Walrasian equilibrium. First, it is *not D-manipulable* (Postlewaite, 1979); that is, an agent cannot be better off by getting rid of part of his endowment. Second, it *does not suffer from the transfer paradox* (Postlewaite and Webb, 1984); that is, the transfer of a portion of his endowment to another individual cannot make an agent better off and the recipient worse off.

The closest contribution to ours is the paper by Pérez-Castrillo and Wettstein (2006). They also provide an ordinal solution in the spirit of the Shapley value for pure exchange economies by extending the idea of Pazner and Schmeidler (1978), who introduce the notion of Pareto-efficient egalitarian equivalent (PEEE) allocations. A PEEE allocation is Pareto efficient and "fair" because, for each agent, it is equivalent preference-wise to the same fixed bundle. Pérez-Castrillo and Wettstein's (2006) ordinal Shapley value (*OSV*) considers possibly different individual endowments and is constructed so that it satisfies "consistency," in the sense that an agent's payoff is based on what he would obtain according to this value when applied to subeconomies.

An essential difference between the *POSh* and the *OSV* is in the domain of the solutions. We consider economies where the consumption bundles are non-negative, whereas the *OSV* is defined in environments where the consumption of a commodity can be positive or negative. Our set-up is more common in the general equilibrium literature and prevents the consumption of a negative amount of goods, such as apples. Let us note that most of the properties of the *POSh*, such as uniqueness, essential single-valueness, empty-agent, and empty-agent out, are not satisfied by the *OSV*.

In addition to Pérez-Castrillo and Wettstein (2006), the early works by Harsanyi (1959), Shapley (1969), and Maschler and Owen (1992) propose extensions of the Shapley value to non-transferable utility environments such as the pure exchange economy that we study. The three proposals are defined in the utility space, and they abstract from the physical environment that generates the utilities. However, as Roemer (1986, 1988) discusses, much information is lost when one moves from the economic environment to the utility space. Thus, on the one hand, these proposals are not ordinal since the solutions are not invariant to alternative representations of the agents' utilities. Moreover, Greenberg et al. (2002) make the observation that the von Neumann and Morgenstein stable sets, defined for the economic environment and the utility space, respectively, may not coincide, even though both are ordinal. On the other hand, as Alon and Lehrer (2019) point out, two very different economic environments, whose solution should be different, may lead to the same allocation of utilities and, hence, the same solution.

McLean and Postlewaite (1989) also extend a notion from the class of TU games to the set of pure exchange economies. They provide an ordinal nucleolus, a solution concept proposed by Schmeidler (1969) for TU games. Nicolò and Perea (2005) and Alon and Lehrer (2019) offer ordinal solutions for bargaining problems.

The remainder of the paper is organized as follows. Section 2 describes the economic environment. It also introduces our new solution concept–the proportional ordinal Shapley solution. Section 3 proves the existence and uniqueness of the *POSh*. Several properties of the *POSh* are also stated and proved. Section 4 considers the environments where the agents have homothetic preferences. Section 5 concludes the paper. All the proofs are in the Appendix.

2 The environment and the solution concept

We consider a *pure exchange economy.* The set of agents is $N \equiv \{1, \ldots, n\}$, with generic agent i. The set of goods is $L \equiv \{1, \ldots, l\}$, which is fixed throughout this paper.

Agent i is described by $(\mathbf{w}_i, \succeq_i)$, where $\mathbf{w}_i \equiv (w_{i1}, \ldots, w_{il}) \in \mathbb{R}^L_+$ is his commodity bundle, and $\succeq_i$ is his preference relation defined over $\mathbb{R}^L_+$. We assume that $\succeq_i$ is reflexive, complete, and transitive for each $i \in N$.[2] We also assume that it is strongly monotone and continuous. Preference $\succeq_i$ is *strongly monotone* if $\mathbf{x} \succ_i \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. Preference $\succeq_i$ is *continuous* if

[2] Agent i's preference $\succeq_i$ is reflexive if $\mathbf{x} \succeq_i \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^L_+$; $\succeq_i$ is complete if either $\mathbf{x} \succeq_i \mathbf{y}$ or $\mathbf{y} \succeq_i \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$; $\succeq_i$ is transitive if $\mathbf{x} \succeq_i \mathbf{y}$ and $\mathbf{y} \succeq_i \mathbf{z}$ imply $\mathbf{x} \succeq_i \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^L_+$.

$\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \succeq_i \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \preceq_i \mathbf{x}\}$ are closed subsets of $\mathbb{R}^L_+$, for all $\mathbf{x} \in \mathbb{R}^L_+$.

A pure exchange economy is a triplet $(N, \mathbf{w}, \succeq)$, where the vector $\mathbf{w}$ is understood as an endowment profile $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ and $\succeq$ is understood as a preference profile $(\succeq_1, \ldots, \succeq_n)$. For a fixed set of agents N, the set of all exchange economies where the agents' preferences are reflexive, complete, transitive, strongly monotone, and continuous is denoted by $\mathcal{E}^N$. The set of all such exchange economies with a finite set of agents is denoted by $\mathcal{E}$.

Definition 1. *A* ***feasible allocation*** *for an exchange economy* $(N, \mathbf{w}, \succeq)$ *is a profile* $\mathbf{z} \equiv (\mathbf{z}_1, \ldots, \mathbf{z}_n) \in \mathbb{R}^{N \times L}_+$ *such that* $\sum_{i \in N} \mathbf{z}_i \leq \sum_{i \in N} \mathbf{w}_i$.

We denote by $Z(N, \mathbf{w})$ the set of feasible allocations for the exchange economy $(N, \mathbf{w}, \succeq)$.

Two feasible allocations are comparable when all agents prefer one to the other in unison. Formally, for $\mathbf{z}, \mathbf{z}' \in Z(N, \mathbf{w})$, we write $\mathbf{z} \succeq \mathbf{z}'$ if $\mathbf{z}_i \succeq_i \mathbf{z}'_i$ for all $i \in N$. Similarly, $\mathbf{z} \sim \mathbf{z}'$ if $\mathbf{z}_i \sim_i \mathbf{z}'_i$ for all $i \in N$. Then, we can define an efficient allocation.

Definition 2. *A feasible allocation* $\mathbf{z} \in Z(N, \mathbf{w})$ *of* $(N, \mathbf{w}, \succeq)$ *is* ***efficient*** *if there is no feasible allocation* $\mathbf{z}' \in Z(N, \mathbf{w})$ *such that* $\mathbf{z}' \succeq \mathbf{z}$ *and* $\mathbf{z}'_j \succ_j \mathbf{z}_j$ *for some* $j \in N$.

We denote by $E(N, \mathbf{w}, \succeq)$ the set of efficient allocations for the exchange economy $(N, \mathbf{w}, \succeq)$.

We now define a solution concept for pure exchange economies.

Definition 3. *A* ***solution*** *is a correspondence* $F : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}^{N \times L}_+$ *such that* $F(N, \mathbf{w}, \succeq) \subseteq Z(N, \mathbf{w})$ *for all* $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Thus, a solution F assigns a set of feasible allocations to each pure exchange economy. Given two solutions F and F', we write $F \subseteq F'$ if $F(N, \mathbf{w}, \succeq) \subseteq F'(N, \mathbf{w}, \succeq)$ for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

A solution F is *single-valued* if F is a function, that is, it prescribes a unique feasible allocation for every economy. A solution F is *essentially single-valued* if $\{\mathbf{y} \in Z(N, \mathbf{w}) \mid \mathbf{y} \sim \mathbf{x}\} = F(N, \mathbf{w}, \succeq)$ for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$ and all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$.[3]

[3] Our definition of essential single-valuedness is formulated for expository convenience. It is stronger than that in the literature (see, e.g., Thomson, 2011), which only requires that every allocation in the solution assigns the same welfare to every agent.

Thus, an essentially single-valued solution prescribes a $\sim$-equivalence class within the set of all feasible allocations. For an essentially single-valued solution F, we write $F_i(N, \mathbf{w}, \succeq) \succeq_i F_i(N, \mathbf{w}', \succeq)$ for $i \in N$ if player i prefers the profiles in $F_i(N, \mathbf{w}, \succeq)$ to the profiles in $F_i(N, \mathbf{w}', \succeq)$. We write $F(N, \mathbf{w}, \succeq) \succeq F(N, \mathbf{w}', \succeq)$ similarly.

Given that agents have initial private endowments, a reasonable solution should ensure that an agent has an incentive to participate instead of walking away with his endowment. The individual rationality of a solution captures this notion:

Definition 4. *A solution F satisfies* ***individual rationality*** *if $\mathbf{x} \succeq \mathbf{w}$ for all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

Next, we formulate two properties that adapt the ideas of the null player property and the null player out property (Derks and Haller, 1999) to pure exchange economies. We identify a type of agents in pure exchange economies who play a similar role as the null players in coalitional games. They are empty-basket agents; we call them empty agents. An agent $i \in N$ is an *empty agent* in the economy $(N, \mathbf{w}, \succeq)$ if $\mathbf{w}_i = \mathbf{0}$. An economy consisting of empty agents only is called an *empty economy.*

The definition of the second property requires the following notation. Let $\mathbf{x} \in \mathbb{R}_+^{N \times L}$ be an allocation profile. Then, for $N' \subseteq N$, we denote by $\mathbf{x}\,|_{N'} \in \mathbb{R}_+^{N' \times L}$ the profile $\mathbf{x}$ restricted to N', that is, $(\mathbf{x}\,|_{N'})_i = \mathbf{x}_i$ for all $i \in N'$. The restrictions of the preference profile are denoted analogously.

Definition 5. *A solution F satisfies the* ***empty-agent property*** *if $\mathbf{x}_i = \mathbf{0}$ for each empty agent $i \in N$ in $(N, \mathbf{w}, \succeq)$, all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

Definition 6. *A solution F satisfies the* ***empty-agent out property*** *if $\mathbf{x}\,|_{N\setminus\{i\}} \in F(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ for each empty agent $i \in N$ in $(N, \mathbf{w}, \succeq)$, all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

The empty-agent and the empty-agent out properties are normative properties. The first one requires that an empty agent be entitled to a zero bundle in any allocation of the solution. In contrast, the empty-agent out property requires that the presence of an empty agent should not influence the allocation of the solution to the rest of the agents. In general, the two properties are logically independent of each other. But, in the presence of efficiency, the empty-agent out property implies the empty-agent property.[4]

It is worth mentioning that Shafer's (1980) example demonstrates that neither the empty-agent property nor the empty-agent out property is satisfied by Shapley's (1969) NTU value.

We now turn to the properties of anonymity and neutrality. The first refers to the agents and the second to the commodities. Before defining the property of anonymity, we introduce the notation for bijections of agents and economies.

Consider an economy $(N, \mathbf{w}, \succeq)$ and a bijection $\pi : N \to N'$. For a feasible allocation $\mathbf{z} \in Z(N, \mathbf{w})$, we define the allocation $\pi\mathbf{z} \in Z(N', \mathbf{w})$ by $(\pi\mathbf{z})_{\pi(i)} \equiv \mathbf{z}_i$ for all $i \in N$. For a preference profile $\succeq$ for N, we define, in a similar fashion, the preference profile $\succeq^\pi$ for N' by $\succeq^\pi_{\pi(i)} = \succeq_i$ for all $i \in N$. Then, for each economy $(N, \mathbf{w}, \succeq)$ and each bijection π of the set of agents, we denote the bijection of the economy by $\pi(N, \mathbf{w}, \succeq) \equiv (\pi[N], \pi\mathbf{w}, \succeq^\pi)$. That is, the structure of economy $\pi(N, \mathbf{w}, \succeq)$ is identical to $(N, \mathbf{w}, \succeq)$, but the names of the agents are changed according to π.

A solution is anonymous if the allocations that it prescribes for an economy are not influenced by the name of the agents. Formally:

Definition 7. *A solution F is* ***anonymous*** *if $\pi\mathbf{x} \in F(\pi(N, \mathbf{w}, \succeq))$ for each bijection $\pi : N \to N'$ and each $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$.*

The property of neutrality, which refers to the names of the commodities, can be defined analogously. For a bijection $\rho : L \to L'$ and a commodity bundle $\mathbf{x} \in \mathbb{R}^L_+$, we define the commodity bundle $\rho\mathbf{x} \in \mathbb{R}^{L'}_+$ by $(\rho x)_{\rho(h)} \equiv x_h$ for all $h \in L$. Also, for a preference profile $\succeq$ over $\mathbb{R}^L_+$, the preference profile $\succeq^\rho$ is defined over $\mathbb{R}^{L'}_+$ by $\rho\mathbf{x} \succeq^\rho_i \rho\mathbf{y}$ if $\mathbf{x} \succeq_i \mathbf{y}$, for all $i \in N$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$. Then, for each economy $(N, \mathbf{w}, \succeq)$ and each bijection ρ, we denote the bijection of the economy by

[4] To see this implication, consider an efficient solution that satisfies the empty-agent out property but does not satisfy the empty-agent property. Then there exists $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ such that $\mathbf{x}_i \neq \mathbf{0}$ for some empty agent i in $(N, \mathbf{w}, \succeq)$. By the empty-agent out property, $\mathbf{x}\mid_{N\setminus\{i\}} \in F(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}})$. Then we could construct a feasible profile $\mathbf{y} \in Z(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}})$ where $\mathbf{y}_j \equiv \mathbf{x}_j + \frac{\mathbf{x}_i}{n-1}$, which would be strictly preferred by every $j \in N\setminus\{i\}$ by strong monotonicity.

$\rho(N, \mathbf{w}, \succeq) \equiv (N, \rho\mathbf{w}, \succeq^{\rho})$. Thus, the structure of economy $\rho(N, \mathbf{w}, \succeq)$ is identical to $(N, \mathbf{w}, \succeq)$, but the names of the commodities are changed according to ρ.

Definition 8. *A solution F is* ***neutral*** *if $\rho\mathbf{x} \in F(\rho(N, \mathbf{w}, \succeq))$ for each bijection $\rho : L \to L'$ and each $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$.*

The last two properties that we propose concern the possibility for an agent to "manipulate" the solution outcome via his endowment. Aumann and Peleg (1974) demonstrate that before the opening of trade, an agent may be better off by getting rid of part of his endowment. In light of this peculiarity, Postlewaite (1979) formulates the following property, which is not implied by efficiency and individual rationality:

Definition 9. *An essentially single-valued solution F is* ***D-manipulable*** *if there exist $\mathbf{w}, \mathbf{w}' \in \mathbb{R}_+^{N \times L}$ such that $\mathbf{w}_i \geq \mathbf{w}'_i$ for some $i \in N$, $\mathbf{w}_j = \mathbf{w}'_j$ for each $j \in N \backslash \{i\}$, and $F_i(N, \mathbf{w}, \succeq) \prec_i F_i(N, \mathbf{w}', \succeq)$.*

An anomaly closely related to D-manipulability is the transfer paradox: a transfer of a portion of his endowment makes the donor better off and the recipient worse off (see, e.g., Postlewaite and Webb, 1984). Definition 10 formally states this paradox.

Definition 10. *An essentially single-valued solution F exhibits the* ***transfer paradox*** *if there exist $\mathbf{w}, \mathbf{w}' \in \mathbb{R}_+^{N \times L}$ and two distinct agents $i, j \in N$ such that $\mathbf{w}_i \geq \mathbf{w}'_i$, $\mathbf{w}_i + \mathbf{w}_j = \mathbf{w}'_i + \mathbf{w}'_j$ and $\mathbf{w}_k = \mathbf{w}'_k$ for each $k \in N \backslash \{i, j\}$, $F_i(N, \mathbf{w}, \succeq) \prec_i F_i(N, \mathbf{w}', \succeq)$, and $F_j(N, \mathbf{w}, \succeq) \succ_i F_j(N, \mathbf{w}', \succeq)$.*

Now we present our solution concept: the *proportional ordinal Shapley* (*POSh*) *solution*. We define the *POSh* in terms of agents' preferences directly. Thus, it is an ordinal solution.

To define the *POSh*, we first define a proportional ordinal potential (a *potential*, for short) in our economic environment, by adapting the idea of the potential introduced by Hart and Mas-Colell (1989) in TU games. In this class of games, a potential is a function that associates a single number to every n-person game. Once we have such a potential function, we can associate to each agent i in the TU game

(N, v) his marginal contribution to the potential, that is, the difference between the potential of (N, v) and the potential of the game $(N \setminus \{i\}, v\mid_{N\setminus\{i\}})$. Then, it is also reasonable to request that the sum of these agents' marginal contributions be efficient, in the sense that it must be equal to the worth of the grand coalition. Hart and Mas-Colell (1989) show that there exists only one such potential function, and the vector of its marginal contributions corresponds to the Shapley value.

In our set of exchange economies, a potential function also associates a single number to each economy. To assign a surplus (an allocation) to each agent i in the economy $(N, \mathbf{w}, \succeq)$ based on the potential, we need a yardstick. We choose agent i's initial endowment $\mathbf{w}_i$ as the reference to measure agent i's welfare. Following Hart and Mas-Colell (1989), we measure his marginal contribution in terms of the difference between the potentials of the economies with and without him, $(N, \mathbf{w}, \succeq)$ and $(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}})$, times his initial endowment. Finally, we require that it should be possible to allocate to each agent a bundle equivalent for him to the bundle corresponding to his marginal contribution to the potential and that this allocation is efficient.

Thus, for the set of exchange economies, we define a potential function as follows:

Definition 11. *A (proportional ordinal)* ***potential function*** $P : \mathcal{E} \to \mathbb{R}_+$ *is defined inductively on the number of players* $|N|$*:*

1. $P(\varnothing) \equiv 0$*;*

2. *for* $(N, \mathbf{w}, \succeq) \in \mathcal{E}$*,* $P(N, \mathbf{w}, \succeq)$ *satisfies that there exists* $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ *such that* $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}})\big)\mathbf{w}_i \sim_i \mathbf{x}_i$ *for all* $i \in N$*.*[5]

The prescription of the $POSh$ is intertwined with our definition of a potential. An allocation is in the $POSh$ if it is efficient and each agent i is indifferent between his prescribed bundle and some multiple of his endowment, where the multiple is equal to the change of potential resulting from his entrance. Thus, we have the following definition of a proportional ordinal Shapley solution in terms of a potential P.

[5] If $N = \{i\}$, we let $P(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}}) \equiv P(\varnothing)$.

Definition 12. *Given a potential function P, a* ***proportional ordinal Shapley solution*** $POSh : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N\times L}$ *is defined by* $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$ *if* $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ *and* $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i \sim_i \mathbf{x}_i$ *for all* $i \in N$.

As we will see in the next section, the $POSh$ is an appealing solution concept. It enjoys properties that echo the properties of the Shapley value, such as efficiency, anonymity, the empty-agent property, and the empty-agent out property. Moreover, it is also immune to well-known anomalies of the Walrasian equilibrium, such as the D-manipulability and the transfer paradox.

At last, the $POSh$ is often easy to compute owing to its neat definition in terms of the potential. For illustration, we compute the $POSh$ for a simple 3-agent economy in Example 1.

Example 1. *Consider the economy with* $L = \{1, 2\}$, $N = \{1, 2, 3\}$, $\mathbf{w}_1 = \mathbf{w}_2 = (4, 4)$, $\mathbf{w}_3 = (2, 2)$, $u_1(x_1, y_1) = 4x_1+y_1$, $u_2(x_2, y_2) = x_2+4y_2$, *and* $u_3(x_3, y_3) = x_3y_3$.

To compute the $POSh(N, \mathbf{w}, u)$, *we need to find the potential of each subeconomy. First, it is easy to see that* $P(\{i\}) = 1$ *for* $i = 1, 2, 3$.[6] *Second, for the subeconomy* $(\{1, 2\}, (\mathbf{w}_1, \mathbf{w}_2), (u_1, u_2))$, *an efficient allocation where both agents obtain an allocation equivalent to* $(P(\{1, 2\}) - 1)(4, 4)$ *assigns the eight units of the first commodity to agent* 1 *and those of the second commodity to agent* 2. *Hence,* $(P(\{1, 2\}) - 1)(4, 4) \sim_1 (8, 0)$ *(and* $(P(\{1, 2\}) - 1)(4, 4) \sim_2 (0, 8)$*), which implies that* $P(\{1, 2\}) = \frac{13}{5}$.

Third, any interior efficient allocation in the subeconomy $(\{1, 3\}, (\mathbf{w}_1, \mathbf{w}_3), (u_1, u_3))$ *satisfies that* $y_3 = 4x_3$. *Therefore, we can conjecture that an efficient allocation in the POSh is* $((6 - x_3, 6 - 4x_3), (x_3, 4x_3))$ *such that* $0 \le x_3 \le \frac{6}{4}$. *Then* $(P(\{1, 3\}) - 1)(4, 4) \sim_1 (6-x_3, 6-4x_3)$ *and* $(P(\{1, 3\})-1)(2, 2) \sim_3 (x_3, 4x_3)$, *that is,* $20(P(\{1, 3\})-1) = 30 - 8x_3$ *and* $4(P(\{1, 3\}) - 1)^2 = 4x_3^2$. *Hence,* $P(\{1, 3\}) = \frac{29}{14}$. *Similarly,* $P(\{2, 3\}) = \frac{29}{14}$ *too.*

Finally, consider the economy $(N, \mathbf{w}, u)$. *We can conjecture that a generic efficient allocation in the POSh must satisfy* $x_1 = y_2$ *and* $x_2 = y_1 = 0$, *that is, it must be* $((x_1, 0), (0, x_1), (10-x_1, 10-x_1))$, *for* $x_1 \in [0, 10]$. *Then,* $(P(N)-P(\{2, 3\}))(4, 4) \sim_1$

[6] In this example, we write $P(\{i\})$ instead of $P(\{i\}, \mathbf{w}_i, \succeq_i)$, and similarly for the other subeconomies, for simplicity.

$(x_1, 0)$ *and* $(P(N) - P(\{1, 2\}))(2, 2) \sim_3 (10 - x_1, 10 - x_1))$. *This system of equations leads to* $P(N) = \frac{1789}{490}$ *and* $x_1 = \frac{387}{49}$. *Therefore, the unique bundle in the POSh is* $((\frac{387}{49}, 0), (0, \frac{387}{49}), (\frac{103}{49}, \frac{103}{49}))$.

Example 1 allows us to make two remarks concerning first, the relationship between the *POSh* and the Walrasian allocations, and second, the property of population monotonicity.

Remark 1. *It is easy to see that the Walrasian equilibrium allocation and the core for Example 1 coincide, which is* $((8, 0), (0, 8), (2, 2))$ *(the Walrasian equilibrium price is* $(1, 1)$*). Therefore, the POSh may not be in the core.*

Remark 2. *In Example 1, the addition of agent 3 to the economy* $(\{1, 2\}, (\mathbf{w}_1, \mathbf{w}_2), (\succeq_1, \succeq_2))$ *harms agents 1 and 2 under POSh by decreasing their utilities from* 8 *to* $\frac{387}{49}$. *That is, the example demonstrates that POSh is not population monotonic in the sense of Sprumont (1990) and Chambers and Hayashi (2020).*

3 Existence and properties of the proportional ordinal Shapley solution

In this section, we establish the existence, uniqueness, and other properties of the proportional ordinal Shapley solution.

To show the existence and uniqueness of the *POSh*, we first state these properties for pure exchange economies where all the agents have a positive vector of endowments. Then we will relate the *POSh* of any pure exchange economy and the *POSh* of the same economy but without its empty agents. This link is possible because, as Proposition 2 will state, any *POSh* solution treats the empty agents as if they would not participate in the economy; that is, any *POSh* satisfies the empty-agent and the empty-agent out properties.

We proceed to analyze the *POSh* in economies without empty agents. Given that any *POSh* solution is based on a potential, we first establish in Proposition 1 the existence and uniqueness of the potential function restricted to economies in which each agent is not empty. Denote by $\mathcal{E}'$ the set of all economies with only non-empty agents.

Proposition 1. *There exists a unique potential function restricted to $\mathcal{E}'$.*

We make two remarks concerning the hypotheses that we use in the proposition.

Remark 3. *We state Proposition 1 for economies where the agents' preferences satisfy strong monotonicity. We cannot replace this hypothesis by the weaker axiom of strict monotonicity. Recall that player i's preference over commodities $\succeq_i$ is* ***strictly monotone*** *if $\mathbf{x} \succ_i \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ such that $x_h > y_h$ for all $h \in L$. To see that this weaker property does not suffice, consider the two-agent economy $(\{1,2\}, \mathbf{w}, \succeq)$, where $\mathbf{w}_1 = (1,1)$, $\mathbf{w}_2 = (2,1)$, $\succeq_1$ is represented by $u_1(x_1, y_1) = x_1$, and $\succeq_2$ is represented by $u_2(x_2, y_2) = y_2$. Both agents' preferences satisfy strict monotonicity but they do not satisfy strong monotonicity. According to Definition 11, $P(\{1\}, \mathbf{w}_1, \succeq_1) = P(\{2\}, \mathbf{w}_2, \succeq_2) = 1$. Moreover, there exists a unique Pareto efficient allocation for $(\{1,2\}, \mathbf{w}, \succeq)$, which assigns $(3,0)$ to agent 1 and $(0,2)$ to agent 2. But then, a number $P(\{1,2\}, \mathbf{w}, \succeq)$ that would satisfy $(P(\{1,2\}, \mathbf{w}, \succeq) - P(\{2\}, \mathbf{w}_2, \succeq_2))(1,1) \sim_1 (3,0)$ and $(P(\{1,2\}, \mathbf{w}, \succeq) - P(\{1\}, \mathbf{w}_1, \succeq_1))(2,1) \sim_2 (0,2)$ does not exist. Hence, a potential function does not exist for this economy.*

Remark 4. *The full strength of the property of the continuity of preferences is not necessary for Proposition 1 to hold. The proof only requires lower semi-continuity of the preferences, i.e., $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \preceq_i \mathbf{x}\}$ is closed for all $\mathbf{x} \in \mathbb{R}^L_+$ and all $i \in N$.*

The existence and uniqueness of the potential function restricted to $\mathcal{E}'$ leads to the existence and essential single-valuedness of the *POSh* restricted to this set:

Corollary 1. *There exists a unique essentially single-valued proportional ordinal Shapley solution restricted to $\mathcal{E}'$.*

We now consider the economies including empty agents. Given that the potential function and the *POSh* solution exist for economies without empty agents, it is convenient to consider, for each economy, the subeconomy that contains all the non-empty agents of the original economy. Formally, we define the *support of the* *economy* $(N, \mathbf{w}, \succeq)$ as the subeconomy where an agent $i \in N$ participates if and only if $\mathbf{w}_i \neq \mathbf{0}$. The support of the economy $(N, \mathbf{w}, \succeq)$ is denoted by $supp(N, \mathbf{w}, \succeq)$. Similarly, we denote by $0(N, \mathbf{w}, \succeq)$ the subeconomy of $(N, \mathbf{w}, \succeq)$ that contains all the empty agents. Thus, each economy $(N, \mathbf{w}, \succeq)$ can be decomposed into two disjoint subeconomies: $supp(N, \mathbf{w}, \succeq)$ and $0(N, \mathbf{w}, \succeq)$.

Using the notion of the support of an economy, it is natural to propose an extension of the potential function to the unrestricted domain as follows: the potential of any economy is equal to the potential of its support plus the potential of the subeconomy containing its empty agents. Proposition 2 uses that the potential of an economy always satisfies the previous relation (see the proof of Proposition 2) to state that, in any *POSh* solution, empty agents receive an empty basket and they do not influence the allocation received by the other agents.

Proposition 2. *Any proportional ordinal Shapley solution in $\mathcal{E}$ satisfies the empty-agent property and the empty-agent out property.*

Proposition 2 highlights that, in a *POSh* solution, empty agents do not obtain any surplus (since they do not contribute to it), and they do not influence the sharing of the surplus allocated to the rest of the agents. It indicates that an empty agent can be viewed as a placeholder under any *POSh*.

Every proportional ordinal Shapley solution satisfies the empty-agent and the empty-agent out properties. Hence, its prescription for agents in a general economy can distinguish between empty and non-empty agents. On the one hand, an empty agent is prescribed a zero bundle by the empty-agent property. On the other hand, a non-empty agent has prescribed a bundle equal to some bundle prescribed by the *POSh* for the support of this economy by the empty-agent out property. Hence, we deduce the uniqueness of the *POSh* for the unrestricted domain from the uniqueness of the *POSh* for the economies without any empty agents. We state the existence and uniqueness of the *POSh* in the unrestricted domain in Theorem 1.

Theorem 1. *There exists a unique essentially single-valued proportional ordinal Shapley solution in $\mathcal{E}$.*

From here onward, we will refer to *the* proportional ordinal Shapley solution since it is unique.

The proof of Theorem 1 proposes to use the following potential function: the potential of an economy is equal to the potential of the support of that economy. That is, the proposed potential assigns a value 0 to any economy consisting of empty agents only. The essentially single-valued $POSh$ is associated with this potential function. However, the uniqueness of the potential for the set of economies without empty agents (Proposition 1) does not extend to the unrestricted domain. In particular, for any empty economy, the potential of each subeconomy can be assigned an arbitrary positive number.

We recall that Theorem 1 establishes the existence and uniqueness of the proportional ordinal Shapley solution for pure exchange economies where preferences are (in addition to reflexive, complete, and transitive) continuous and strongly monotone. The requirements for the existence of the $POSh$ are incomparable with those for Walrasian equilibrium. Indeed, the existence of Walrasian equilibrium requires each agent's preference to be continuous, convex, and non-satiated, and each agent's endowment strictly positive (see Border, 2017). On the one hand, strong monotonicity is a stronger assumption than non-satiation. On the other hand, neither convex preferences nor strictly positive endowment is needed for the existence of the $POSh$.

We have defined the $POSh$ using the idea of the potential, which characterizes the Shapley value. We have seen that the $POSh$ exists and is unique and essentially single-valued. We now prove and discuss other properties related to properties that the Shapley value satisfies in TU games.

The classic characterization of the Shapley value in TU games uses efficiency, null player, anonymity, and linearity. By definition, the $POSh$ is an efficient solution. Moreover, it satisfies the empty-agent property (Proposition 2), which corresponds in our pure exchange economy to the null player axiom in TU games. In fact, it also satisfies the empty-agent out property that, under efficiency, is stronger than the empty-agent property.

We now turn to the axiom of anonymity. Proposition 3 states that the $POSh$ satisfies not only the property of anonymity but also neutrality. That is, it is immune to changes in the names of the agents and commodities.

Proposition 3. *The proportional ordinal Shapley solution satisfies anonymity and neutrality in $\mathcal{E}$.*

Concerning the last axiom in the classic characterization of the Shapley value in TU games, the $POSh$ does not satisfy a property in the spirit of the linearity axiom of the Shapley value. When preferences are representable by a quasilinear utility function, it generally leads to a different level of utility than the Shapley value of the associated TU game. In the next section, we identify the TU value corresponding to the $POSh$ when agents' preferences are quasilinear and homothetic.

We now turn to characterize the $POSh$ using an idea similar to the "coalitional dividends" (Harsanyi, 1959). Harsanyi's characterization of the Shapley value is very different from the classic characterization. His approach considers that every coalition negotiates a vector of dividends such that the sum of all coalitional dividends vectors constitute a feasible allocation for the grand coalition. Therefore, the dividends that a coalition allocates are what is left after all its proper subcoalitions have received their corresponding dividends. Proposition 4 shows that the $POSh$ solution can be characterized in a similar manner. We can construct a system of dividends that leads to the following alternative representation of the $POSh$:

Proposition 4. *For all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$, there exists a vector of dividends $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ such that for all $N' \in 2^N \setminus \{\varnothing\}$, $\mathbf{x} \in POSh(N', \mathbf{w}\,|_{N'}, \succeq|_{N'})$ if and only if $\mathbf{x} \in E(N', \mathbf{w}\,|_{N'}, \succeq|_{N'})$ and $\big(\sum_{\substack{T \ni i \\ T \subseteq N'}} d_T\big)\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in N'$.*

We note that Pérez-Castrillo and Wettstein (2006) also provide a characterization of their ordinal Shapley value (OSV) in terms of dividends. However, there is an important difference between their characterization and ours. For the OSV, the dividends d_S and d'_S of the same coalition $S \subseteq N'$ for an economy $(N, \mathbf{w}, \succeq)$ and its subeconomy $(N', \mathbf{w}\,|_{N'}, \succeq|_{N'})$, respectively, may be different. By contrast, for the $POSh$, the dividends of the same coalition of an economy and its subeconomy must be the same. In this sense, our characterization is closer in spirit to Harsanyi's (1959) characterization of the Shapley value in the set of TU games.

The last part of the section provides three additional properties of the $POSh$, which are relevant in the pure exchange economies where it is defined. First, we show that, as the Walrasian equilibrium, the $POSh$ is individually rational; that is, no agent has an incentive to refuse to participate in the solution.

Proposition 5. *The proportional ordinal Shapley solution satisfies individual rationality in* $\mathcal{E}$.

An immediate consequence of the individual rationality of the $POSh$, together with its essential single-valuedness, is that a version of the second fundamental welfare theorem holds for the $POSh$: any efficient allocation is sustained under the $POSh$ solution.

Corollary 2. *If* $\mathbf{w}$ *is efficient in* $(N, \mathbf{w}, \succeq)$*, then* $\mathbf{w} \in POSh(N, \mathbf{w}, \succeq)$.

Finally, we highlight that, in contrast to the Walrasian equilibrium, the $POSh$ is robust against agents' manipulation of their initial endowment. Proposition 6 shows that an agent never has an incentive to throw away any part of his initial endowment, that is, the $POSh$ is not D-manipulable. Proposition 7 states that the $POSh$ does not exhibit the transfer paradox; that is, transferring a portion of his initial endowment cannot make the donor better of and the recipient worse off.

Proposition 6. *The proportional ordinal Shapley solution is not D-manipulable in* $\mathcal{E}$.

Proposition 7. *The proportional ordinal Shapley solution does not exhibit the transfer paradox in* $\mathcal{E}$.

4 The proportional ordinal Shapley solution when agents have homothetic preferences

This section restricts attention to the set of exchange economies where agents have homothetic preferences to provide another characterization of the $POSh$ solution. It also relates the $POSh$ to a Shapley-like solution concept in transferable utility games. We recall that an agent's preference $\succeq$ is **homothetic** if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ and all $\alpha \in \mathbb{R}_+$, $\mathbf{x} \succeq \mathbf{y}$ if and only if $\alpha\mathbf{x} \succeq \alpha\mathbf{y}$. We denote by $\mathcal{E}^h$ the set of economies where agents' preferences are homothetic.

In the set $\mathcal{E}^h$, we characterize the *POSh* solution using a "weighted balanced contributions" axiom. Myerson (1980) proposes the fairness notion that "any two players should enjoy the same gains from their cooperation together, relative to what they would get without cooperation." Following this idea, he defines the balanced contributions axiom and shows that, together with efficiency, it characterizes the Shapley value. Hart and Mas-Colell (1989) extend this result to weighted Shapley values, using a weighted balanced contributions axiom.

Proposition 8 shows that the *POSh* solution is also characterized by a weighted balanced contributions axiom, where the level of his initial endowment gives the weight associated with each agent.

Proposition 8. *Consider a solution F for the set of economies where agents have homothetic preferences, that is, $F : \mathcal{E}^h \rightsquigarrow \bigcup_N \mathbb{R}^{N\times L}_+$. Then, $F = POSh$ if and only if $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and*

$$\frac{u_i(\mathbf{x}_i^N) - u_i(\mathbf{x}_i^{N\setminus\{j\}})}{u_i(\mathbf{w}_i)} = \frac{u_j(\mathbf{x}_j^N) - u_j(\mathbf{x}_j^{N\setminus\{i\}})}{u_j(\mathbf{w}_j)}, \tag{1}$$

for any $\mathbf{x}^N \in F(N, \mathbf{w}, \succeq)$, $\mathbf{x}^{N\setminus\{j\}} \in F(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})$, $\mathbf{x}^{N\setminus\{i\}} \in F(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$, any $(N, \mathbf{w}, \succeq) \in \mathcal{E}^h$, and any profile of utility functions $(u_i)_{i\in N}$ that represents the preference profile $\succeq$.[7]

The interpretation of the weighted balanced contribution property that characterizes the *POSh* in homothetic environments is the same as in TU games. Consider any representation of the agents' preferences. Then, the difference in the utility that any agent obtains in the economies with him and without him, normalized by the utility that he derives from his initial endowment, must be the same for all the agents. It is a fairness requirement that takes into account that an agent's contribution to the common pool is more significant when his initial endowment is larger.

Interestingly, given that we use the initial endowments as the yardstick to measure the variation of an agent's welfare in the *POSh*, we can connect the *POSh*

[7] We normalize $u_i(\mathbf{0}) = 0$ for all $i \in N$.

and the "proportional Shapley value" (PSh for short) for TU games (Besner, 2016, and Béal at al., 2018), when preferences are homothetic. The PSh is a weighted Shapley value in which the players' weights are endogenously given by the players' stand-alone worths. Béal et al. (2018) use Hart and Mas-Colell's (1989) result to characterize the PSh by efficiency and the "proportional balanced contributions" property, which is the weighted balanced contributions property with weights proportional to the players' stand-alone worth. Thus, the proportional balanced contributions property is similar to the property stated in (1) once we consider $u_i(\mathbf{w}_i)$ as the utility worth of agent i if he stays alone.

Moreover, the $POSh$ solution and the PSh coincide in pure exchange economies $(N, \mathbf{w}, \succeq)$ in which each agent i's preference $\succeq_i$ is representable as a (homothetic and) quasi-linear utility function $u_i(\mathbf{x}) = w_i(\mathbf{x} \mid_{L\setminus\{m\}}) + x_m$, where $\mathbf{x} \mid_{L\setminus\{m\}} \in \mathbb{R}_+^{L\setminus\{m\}}$ and $x_m \in \mathbb{R}$. The specially treated commodity $m \in L$ in each agent's utility function can be interpreted as "money." Such an economy can be naturally turned into a TU game (N, v) by letting $v(S) \equiv \max_{\mathbf{z}\in Z(S,\mathbf{w}|_S)} \sum_{i\in S} u_i(\mathbf{z}_i)$ for each $S \in 2^N \setminus \{\varnothing\}$. Then, the utility profile of any allocation in $POSh(N, \mathbf{w}, \succeq)$ according to u coincides with $PSh(N, v)$.[8]

5 Conclusion

We espouse a new ordinal solution concept for pure exchange economies, the $POSh$ solution. Its construction is inspired by the potential function, which allows a nice characterization of the Shapley value in TU games. The $POSh$ solution satisfies properties similar to the Shapley value, such as efficiency, anonymity, and properties related to null players. It is also individually rational and does not suffer from agents' manipulation via their initial endowment.

We further highlight the link between the $POSh$ for pure exchange economies and the Shapley value for TU games by characterizing the $POSh$ through Harsanyi's (1959) coalitional dividends and, when the agents have homothetic preferences, through a weighted balanced contribution property à la Myerson (1980).

[8] See the Appendix for the proof of this result.

One natural avenue for future research is extending our solution concept and its properties to pure exchange economies with a continuum of agents of finite types. It is easy to extend the notions of the potential and the proportional ordinal Shapley solution to these economies. However, the analysis of the properties of the $POSh$ in these environments is outside the scope of this paper.

Appendix

Proof of Proposition 1. First, we show that there exists at most one potential function. Suppose otherwise, that is, suppose that there exist two distinct potential functions P and P'. Then, without loss of generality, assume that for $(N, \mathbf{w}, \succeq)$, it happens that $P(N, \mathbf{w}, \succeq) > P'(N, \mathbf{w}, \succeq)$ and $P(S, \mathbf{w}|_S, \succeq|_S) = P'(S, \mathbf{w}|_S, \succeq|_S)$ for all $S \in 2^N \setminus \{N\}$. This implies that there exist two allocations $\mathbf{x}, \mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{x}_k \sim_k (P(N, \mathbf{w}, \succeq) - P(N \setminus \{k\}, \mathbf{w}))\mathbf{w}_k \succ_k (P'(N, \mathbf{w}, \succeq) - P'(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{k\}}, \succeq|_{N\setminus\{k\}}))\mathbf{w}_k \sim_k \mathbf{y}_k$ for all $k \in N$, where the strict preference follows from strong monotonicity and the premise on P and P'. However, this contradicts that $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$. Therefore, there exists at most one potential function.

Second, to prove the existence of a potential function for any $(N, \mathbf{w}, \succeq) \in \mathcal{E}'$, we construct inductively the potential $P(S, \mathbf{w}|_S, \succeq|_S)$ of each subeconomy $(S, \mathbf{w}|_S, \succeq|_S)$ of $(N, \mathbf{w}, \succeq) \in \mathcal{E}'$, on the number of agents $|S|$:

1. For $|S| = 0$, $P(\varnothing) \equiv 0$;

2. for $|S| \geq 1$, we hypothesize that $P(T, \mathbf{w}|_T, \succeq|_T)$ has been defined for each $T \subsetneq S$. Then, we define $P(S, \mathbf{w}|_S, \succeq|_S) \equiv \sup\{P \in \mathbb{R} \mid \exists \mathbf{x} \in Z(S, \mathbf{w}|_S)$ such that $\big(P - P(S \setminus \{i\}, \mathbf{w}|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \preceq_i \mathbf{x}_i$ for all $i \in S\}$.

It is easy to see that $P(S, \mathbf{w}|_S, \succeq|_S) = 1$ for $|S| = 1$. Consider S with $|S| \geq 2$ (and the induction hypothesis stated in point 2 above). We start by showing that $P(S, \mathbf{w}|_S, \succeq|_S)$ is well-defined for $|S| \geq 2$, i.e., the set $\mathit{\Pi} \equiv \{P \in \mathbb{R} \mid \exists \mathbf{x} \in Z(S, \mathbf{w}|_S)$ such that $\big(P - P(S \setminus \{i\}, \mathbf{w}|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \preceq_i \mathbf{x}_i$ for all $i \in S\}$ is not empty.

Let $k \in \operatorname{argmax}_{i\in S} P(S \setminus \{i\}, \mathbf{w}|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})$. We claim that $P(S \setminus \{k\}, \mathbf{w}|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}}) \in \mathit{\Pi}$. To show it, take $\mathbf{y} \in \mathbb{R}_+^{S\times L}$ such that $\mathbf{y}_k = \mathbf{0}$ and $\mathbf{y}|_{S\setminus\{k\}} \in POSh(S\setminus$

$\{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}})$. It is immediate that $\mathbf{y} \in Z(S, \mathbf{w}\,|_S)$. Moreover, $\big(P(S \setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}}) - P(S \setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}})\big)\mathbf{w}_k \sim_i \mathbf{y}_k = 0$. Lastly, for any $i \in S \setminus \{k\}$, $\big(P(S \setminus \{k\}, \mathbf{w}|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}}) - P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \preceq_i \big(P(S \setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}}) - P(S \setminus \{i,k\}, \mathbf{w}\,|_{S\setminus\{i,k\}}, \succeq|_{S\setminus\{i,k\}})\big)\mathbf{w}_i \sim_i \mathbf{y}_i$ because $P(S\setminus\{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}}) \geq P(S\setminus\{i,k\}, \mathbf{w}\,|_{S\setminus\{i,k\}}, \succeq|_{S\setminus\{i,k\}})$ and $\mathbf{y}\,|_{S\setminus\{k\}} \in POSh(S\setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}})$.

Once we have proved that $P(S, \mathbf{w}\,|_S, \succeq|_S)$ is well-defined, we show that it satisfies that $\big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in S$ and some $\mathbf{x} \in E(S, \mathbf{w}\,|_S, \succeq|_S)$. Note that $P(S, \mathbf{w}\,|_S, \succeq|_S)$ satisfies that $\big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S\setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \preceq_i \mathbf{x}_i$ for all $i \in S$ and some $\mathbf{x} \in E(S, \mathbf{w}\,|_S, \succeq|_S)$ because each agent's preference is continuous and $Z(S, \mathbf{w}\,|_S)$ is closed. We prove our claim by contradiction: if there exists $k \in S$ such that $\big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}})\big)\mathbf{w}_k \prec_k \mathbf{x}_k$, then it is possible to construct an alternative feasible allocation profile $\mathbf{y} \in Z(S, \mathbf{w}\,|_S)$ such that $\big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \prec_i \mathbf{y}_i$ for all $i \in S$. The existence of the profile $\mathbf{y}$ would imply that the supremum was not attained at $P(S, \mathbf{w}\,|_S, \succeq|_S)$ since $P(S, \mathbf{w}\,|_S, \succeq|_S)$ could be increased by a sufficiently small amount without violating feasibility. To construct $\mathbf{y}$ from $\mathbf{x}$, first note that $\mathbf{0} \preceq_k \big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}})\big)\mathbf{w}_k \prec_k \mathbf{x}_k$, hence $x_{kh} > 0$ for some $h \in L$. Define $\mathbf{y}$ by

$$y_{ig} \equiv \begin{cases} x_{ig} & \text{if } i \in S \text{ and } g \in L \setminus \{h\}, \\ x_{ig} - \epsilon & \text{if } i = k \text{ and } g = h, \\ x_{ig} + \frac{\epsilon}{|S|} & \text{if } i \in S \setminus \{k\} \text{ and } g = h, \end{cases}$$

where $\epsilon \in \mathbb{R}_{++}$ is sufficiently small so that $\big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{k\}, \mathbf{w}\,|_{S\setminus\{k\}}, \succeq|_{S\setminus\{k\}})\big)\mathbf{w}_k \prec_k \mathbf{y}_k$ and $y_{kh} \geq 0$. By strong monotonicity, we have $\big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i \prec_i \mathbf{y}_i$ for all $i \in S$.

Therefore, we have proven the existence of a potential function restricted to $\mathcal{E}'$, which concludes the proof of the proposition. □

Proof of Proposition 2. The empty-agent property follows from the efficiency included in Definition 12, once we prove the empty-agent out property, which we now do.

First, we claim that any potential function satisfies

$$P(N, \mathbf{w}, \succeq) = P(supp(N, \mathbf{w}, \succeq)) + P(0(N, \mathbf{w}, \succeq)). \tag{2}$$

We prove equation (2) by an induction on p, by which we denote the number of non-empty agents of an economy $(N, \mathbf{w}, \succeq)$ with q empty agents (q is an arbitrary fixed positive number). The equation holds trivially for an economy with only q empty agents, i.e., when $p = 0$. Then, consider an economy $(N, \mathbf{w}, \succeq)$ with $p \geq 1$ non-empty agents and q empty agents. Denote by $\mathbf{x} \in E(supp(N, \mathbf{w}, \succeq))$ an allocation profile satisfying that $\mathbf{x}_i \sim_i \big(P(supp(N, \mathbf{w}, \succeq)) - P(supp(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}))\big)\mathbf{w}_i$ for all non-empty agent i. The allocation $\mathbf{x}$ satisfies that for each non-empty agent i,

$$\begin{aligned}
\mathbf{x}_i \sim_i & \big[\big(P(supp(N, \mathbf{w}, \succeq)) + P(0(N, \mathbf{w}, \succeq))\big) - \big(P(supp(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})) \\
&\quad + P(0(N, \mathbf{w}, \succeq))\big)\big]\mathbf{w}_i \\
= & \big[\big(P(supp(N, \mathbf{w}, \succeq)) + P(0(N, \mathbf{w}, \succeq))\big) - \big(P(supp(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})) \\
&\quad + P(0(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}))\big)\big]\mathbf{w}_i \\
= & \big[\big(P(supp(N, \mathbf{w}, \succeq)) + P(0(N, \mathbf{w}, \succeq))\big) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big]\mathbf{w}_i,
\end{aligned}$$

where the first equality follows from the premise that i is not an empty agent and the second from the induction hypothesis (there exist $p-1$ non-empty agents in $(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$). Then consider a new allocation profile $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$, where $\mathbf{y}_j = \mathbf{x}_j$ for each non-empty agent j and $\mathbf{y}_k = \mathbf{0}$ for each empty agent k. Notice that the constructed profile $\mathbf{y}$ satisfies that $\mathbf{y}_i \sim_i \big(P^o(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i$ for all $i \in N$, where we define $P^o(N, \mathbf{w}, \succeq) \equiv P(supp(N, \mathbf{w}, \succeq)) + P(0(N, \mathbf{w}, \succeq))$. Moreover, by strong monotonicity, $p \geq 1$, and an argument similar to that establishing the uniqueness of the potential function restricted to $\mathcal{E}'$ in Proposition 1, we have that the numerical value of the potential is unique, hence $P^o(N, \mathbf{w}, \succeq) = P(N, \mathbf{w}, \succeq)$. Finally, since q is arbitrary, we have proven the equation (2), which immediately implies the empty-agent out property of any $POSh$. $\square$

Proof of Theorem 1. Let us denote by $POSh'$ the proportional ordinal Shapley solution restricted to $\mathcal{E}'$, which is unique and essentially single-valued by Corollary 1. First, by Proposition 2, any $POSh$ for $\mathcal{E}$ satisfies the empty-agent property and the empty-agent out property. Therefore, for any $POSh$ and any $(N, \mathbf{w}, \succeq)$,

$$\mathbf{x} \in POSh(N, \mathbf{w}, \succeq) \iff \mathbf{x} \mid_S \in POSh'(S, \mathbf{w} \mid_S, \succeq\mid_S) \text{ and } \mathbf{x} \mid_{N \setminus S} = \mathbf{0}, \tag{3}$$

where $(S, \mathbf{w} \mid_S, \succeq\mid_S) = \mathit{supp}(N, \mathbf{w}, \succeq)$. Hence, if $POSh$ exists for $\mathcal{E}$, it is also unique and essentially single-valued.

Second, denote by P' the potential associated with $POSh'$ in $\mathcal{E}'$. We now propose the following potential function $P : \mathcal{E} \to \mathbb{R}$:

$$P(N, \mathbf{w}, \succeq) \equiv P'(\mathit{supp}(N, \mathbf{w}, \succeq)). \tag{4}$$

We show that the function P can be associated with the $POSh$ that we constructed in (3) for $\mathcal{E}$; that is, $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$ and all $i \in N$. If $\mathbf{w}_i = \mathbf{0}$, then the result is immediate because $POSh_i(N, \mathbf{w}, \succeq) = \{\mathbf{0}\}$. Otherwise, consider an economy $(N, \mathbf{w}, \succeq)$ where i is a non-empty agent, and $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$. Then, equation (3) states that $\mathbf{x} \in POSh'_i(\mathit{supp}(N, \mathbf{w}, \succeq))$. Therefore, $\mathbf{x}_i \sim_i \big(P'(\mathit{supp}(N, \mathbf{w}, \succeq)) - P'(\mathit{supp}(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}))\big)\mathbf{w}_i = \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i$.

Finally, let $N' \subseteq N$ be the set of non-empty agents in $(N, \mathbf{w}, \succeq)$. Then $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and $\mathbf{x}_i \equiv \mathbf{0}$ for all $i \in N \setminus N'$ if and only if $\mathbf{x} \mid_{N'} \in E(\mathit{supp}(N, \mathbf{w}, \succeq))$. Thus, the constructed $P : \mathcal{E} \to \mathbb{R}$ is a potential function associated with the $POSh$ for $\mathcal{E}$, which means that there exists a $POSh$ for $\mathcal{E}$. $\square$

Proof of Proposition 3. We first prove anonymity of the $POSh$. It is easy to see that the efficient allocation correspondence is anonymous, that is,

$$\mathbf{x} \in E(N, \mathbf{w}, \succeq) \implies \pi\mathbf{x} \in E(\pi(N, \mathbf{w}, \succeq)), \tag{5}$$

for all bijection $\pi : N \to N'$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Consider an economy $(N, \mathbf{w}, \succeq) \in \mathcal{E}$ and an arbitrary bijection $\pi : N \to N'$. We are going to prove simultaneously that $(\pi \mid_S)\mathbf{x} \in POSh(\pi \mid_S (S, \mathbf{w} \mid_S, \succeq\mid_S))$ for every $\mathbf{x} \in POSh(S, \mathbf{w}|_S, \succeq|_S)$ and every $S \in 2^N \setminus \{\varnothing\}$ and that we could let $P(\pi \mid_S (S, \mathbf{w} \mid_S, \succeq\mid_S)) = P(S, \mathbf{w} \mid_S, \succeq\mid_S)$ for every $S \subseteq N$ by induction on $|S|$. For $|S| = 0$, it trivially holds because $P(S) = P(\varnothing) = 0$. For $|S| \geq 1$, $\mathbf{x} \in POSh(S, \mathbf{w}|_S, \succeq|_S)$, by definition, means that there exists $\mathbf{x} \in E(S, \mathbf{w}|_S, \succeq|_S)$ such that $\mathbf{x}_i \sim_i \big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i$ for all $i \in S$. By the induction hypothesis, $P(\pi\ |_{S\setminus\{i\}}\ (S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})) = P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})$ for all $i \in S$. Then, $\mathbf{x} \in POSh(S, \mathbf{w}|_S, \succeq|_S)$ implies that there exists $(\pi|_S)\mathbf{x} \in E(\pi\,|_S\,(S, \mathbf{w}|_S, \succeq|_S))$ such that $((\pi|_S)\mathbf{x})_{\pi(i)} \sim^{\pi}_{\pi(i)} \big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(\pi\,|_{S\setminus\{i\}}\,(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}}))\big)(\pi\mathbf{w})_{\pi(i)}$ for all $i \in S$ because of (5), $((\pi\,|_S)\mathbf{x})_{\pi(i)} = \mathbf{x}_i$, and $\sim^{\pi}_{\pi(i)} = \sim_i$. That is, there exists $(\pi\,|_S)\mathbf{x} \in E(\pi\,|_S\,(S, \mathbf{w}\,|_S, \succeq|_S))$ such that $(\pi\,|_S\,\mathbf{x})_j \sim^{\pi}_j \big(P(S, \mathbf{w}|_S, \succeq|_S) - P(\pi\,|_{S\setminus\{\pi^{-1}(j)\}})(S \setminus \{\pi^{-1}(j)\}, \mathbf{w}\,|_{S\setminus\{\pi^{-1}(j)\}}, \succeq|_{S\setminus\{\pi^{-1}(j)\}})\big)(\pi\mathbf{w})_j$ for all $j \in \pi[S]$. It means that $(\pi\,|_S)\mathbf{x} \in POSh(\pi\,|_S\,(S, \mathbf{w}\,|_S, \succeq|_S))$ and $P(\pi\,|_S\,(S, \mathbf{w}\,|_S, \succeq|_S)) = P(S, \mathbf{w}\,|_S, \succeq|_S)$, which concludes the inductive step. Hence, $POSh$ satisfies anonymity.

To prove neutrality, again it is easy to see that the efficient allocation correspondence is neutral, that is,

$$\mathbf{x} \in E(N, \mathbf{w}, \succeq) \implies \rho\mathbf{x} \in E(\rho(N, \mathbf{w}, \succeq)), \tag{6}$$

for all bijection $\rho : L \to L'$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Consider an economy $(N, \mathbf{w}, \succeq) \in \mathcal{E}$ and an arbitrary bijection $\rho : L \to L'$. We are going to prove simultaneously that $\rho\mathbf{x} \in POSh(\rho(S, \mathbf{w}\,|_S, \succeq|_S))$ for every $\mathbf{x} \in POSh(S, \mathbf{w}\,|_S, \succeq|_S)$ and every $S \in 2^N \setminus \{\varnothing\}$ and that we could let $P(\rho(S, \mathbf{w}\,|_S, \succeq|_S)) = P(S, \mathbf{w}\,|_S, \succeq|_S)$ for every $S \subseteq N$ by induction on $|S|$. For $|S| = 0$, it trivially holds because $P(S) = P(\varnothing) = 0$. For $|S| \geq 1$, $\mathbf{x} \in POSh(S, \mathbf{w}\,|_S, \succeq|_S)$, by definition, means that there exists $\mathbf{x} \in E(S, \mathbf{w}\,|_S, \succeq|_S)$ such that $\mathbf{x}_i \sim_i \big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big)\mathbf{w}_i$ for all $i \in S$. By the induction hypothesis, $P(\rho(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})) = P(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})$ for all $i \in S$. Then, $\mathbf{x} \in POSh(S, \mathbf{w}\,|_S, \succeq|_S)$ implies that there exists $\rho\mathbf{x} \in E(\rho(S, \mathbf{w}\,|_S, \succeq|_S))$ such that $\rho\mathbf{x}_i \sim^{\rho}_i \big(P(S, \mathbf{w}\,|_S, \succeq|_S) - P(\rho(S \setminus \{i\}, \mathbf{w}\,|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}}))\big)(\rho\mathbf{w})_i$ for all $i \in S$ because of (6) and the definitions of $\rho\mathbf{x}_i$ and $\sim^{\rho}_i$. It means that $\rho\mathbf{x} \in POSh(\rho(S, \mathbf{w}\,|_S, \succeq|_S))$ and $P(\rho(S, \mathbf{w}\,|_S, \succeq|_S)) = P(S, \mathbf{w}\,|_S, \succeq|_S)$, which concludes the inductive step. Hence, the $POSh$ satisfies neutrality. □

Proof of Proposition 4. Consider $(N, \mathbf{w}, \succeq) \in \mathcal{E}$. We define a vector of dividends $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ by

$$d_S \equiv \sum_{T \subseteq S} (-1)^{s+t} P(T, \mathbf{w}\,|_T, \succeq|_T), \tag{7}$$

for all $S \in 2^N \setminus \{\varnothing\}$. Then it is easy to verify that

$$P(S, \mathbf{w}|_S, \succeq|_S) = \sum_{T \in 2^S \setminus \{\varnothing\}} d_T, \tag{8}$$

for all $S \in 2^N \setminus \{\varnothing\}$. Furthermore, it yields

$$P(N', \mathbf{w}|_{N'}, \succeq|_{N'}) - P(N' \setminus \{i\}, \mathbf{w}|_{N' \setminus \{i\}}, \succeq|_{N' \setminus \{i\}}) = \sum_{\substack{T \ni i \\ T \subseteq N'}} d_T, \tag{9}$$

for all $N' \in 2^N \setminus \{\varnothing\}$ and all $i \in N'$.

By substituting equation (9) in Definition 12, we obtain the representation of the $POSh$ in terms of dividends in Proposition 4. □

Proof of Proposition 5. Denote P the potential function associated to the $POSh$. We show that $\mathbf{x}_i \succeq_i \mathbf{w}_i$ for all $i \in N$, all $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$ by induction on $|N|$. Since $POSh(\{i\}, \mathbf{w}_i, \succeq_i) = \{\mathbf{w}_i\}$ for $N = \{i\}$, our assertion trivially holds for $|N| = 1$.

For $|N| \geq 2$, assume that the property holds for any economy with less than $|N|$ agents. Suppose , by contradiction, that it does not hold for $(N, \mathbf{w}, \succeq)$, that is, there exists $i \in N$ such that $\mathbf{x}_i \sim_i \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}})\big)\mathbf{w}_i \prec_i \mathbf{w}_i$. Then there must exist $j \in N \setminus \{i\}$ such that $\mathbf{x}_j \succ_j \big(P(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}}) - P(N \setminus \{i, j\}, \mathbf{w}|_{N \setminus \{i,j\}}, \succeq|_{N \setminus \{i,j\}})\big)\mathbf{w}_j$. The existence of such an agent j follows from $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$, $\mathbf{x}_i \prec_i \mathbf{w}_i$, and the feasibility of the allocation that assigns agent i with $\mathbf{w}_i$ and the rest of agents with a bundle prescribed by $POSh(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}$ $, \succeq|_{N \setminus \{i\}})$, which is individually rational by the induction hypothesis. Therefore, there exists $j \in N \setminus \{i\}$ such that $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{j\}, \mathbf{w}|_{N \setminus \{j\}}, \succeq|_{N \setminus \{j\}})\big)\mathbf{w}_j \sim_j \mathbf{x}_j \succ_j \big(P(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}}) - P(N \setminus \{i, j\}, \mathbf{w}|_{N \setminus \{i,j\}}, \succeq|_{N \setminus \{i,j\}})\big)\mathbf{w}_j$. Agent j's strict preference $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{j\}, \mathbf{w}\, |_{N \setminus \{j\}}, \succeq|_{N \setminus \{j\}})\big)\mathbf{w}_j \succ_j \big(P(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}}) - P(N \setminus \{i, j\}, \mathbf{w}|_{N \setminus \{i,j\}}, \succeq|_{N \setminus \{i,j\}})\big)\mathbf{w}_j$ implies that

$$\begin{aligned} &P(N, \mathbf{w}, \succeq) + P(N \setminus \{i, j\}, \mathbf{w}|_{N \setminus \{i,j\}}, \succeq|_{N \setminus \{i,j\}}) - P(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}}) \\ &- P(N \setminus \{j\}, \mathbf{w}|_{N \setminus \{j\}}, \succeq|_{N \setminus \{j\}}) > 0 \end{aligned} \tag{10}$$

by strong monotonicity.

On the other hand, by the induction hypothesis, we have $\big(P(N\setminus\{j\},\mathbf{w}|_{N\setminus\{j\}},\succeq|_{N\setminus\{j\}}))-P(N\setminus\{i,j\},\mathbf{w}|_{N\setminus\{i,j\}},\succeq|_{N\setminus\{i,j\}}))\big)\mathbf{w}_i\succeq_i\mathbf{w}_i$. Together with the inequality (10), it implies that

$$\begin{aligned}
\mathbf{x}_i \sim_i &\big(P(N,\mathbf{w},\succeq)-P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))\big)\mathbf{w}_i\\
=&\big[\big(P(N\setminus\{j\},\mathbf{w}|_{N\setminus\{j\}},\succeq|_{N\setminus\{j\}}))-P(N\setminus\{i,j\},\mathbf{w}|_{N\setminus\{i,j\}},\succeq|_{N\setminus\{i,j\}}))\big)\\
&+\big(P(N,\mathbf{w},\succeq)+P(N\setminus\{i,j\},\mathbf{w}|_{N\setminus\{i,j\}},\succeq|_{N\setminus\{i,j\}}))-P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))\\
&-P(N\setminus\{j\},\mathbf{w}|_{N\setminus\{j\}},\succeq|_{N\setminus\{j\}}))\big)\big]\mathbf{w}_i\\
\succeq_i&\big(P(N\setminus\{j\},\mathbf{w}|_{N\setminus\{j\}},\succeq|_{N\setminus\{j\}}))-P(N\setminus\{i,j\},\mathbf{w}|_{N\setminus\{i,j\}},\succeq|_{N\setminus\{i,j\}}))\big)\mathbf{w}_i\succeq_i\mathbf{w}_i,
\end{aligned}$$

which contradicts our assumption. Therefore, the $POSh$ satisfies individual rationality. □

Proof of Proposition 6. Consider two economies $(N,\mathbf{w},\succeq),(N,\mathbf{w}',\succeq)\in\mathcal{E}$ such that $\mathbf{w}_i>\mathbf{w}'_i$ for $i\in N$ and $\mathbf{w}_j=\mathbf{w}'_j$ for each $j\in N\setminus\{i\}$. We claim that $P(S,\mathbf{w}|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})\geq P(S,\mathbf{w}'|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}'|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})$ for all $S\subseteq N$ such that $S\ni i$. We prove the claim by induction on $|S|$. It trivially holds for $|S|=1$.

For $S\subseteq N$ such that $|S|\geq 2$, suppose otherwise: $P(S,\mathbf{w}|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})<P(S,\mathbf{w}'|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}'|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})$ for some $i\in S$ and $P(T,\mathbf{w}|_T,\succeq|_T)-P(T\setminus\{j\},\mathbf{w}|_{T\setminus\{j\}},\succeq|_{T\setminus\{j\}})\geq P(T,\mathbf{w}'|_T,\succeq|_T)-P(T\setminus\{j\},\mathbf{w}'|_{T\setminus\{j\}},\succeq|_{T\setminus\{j\}})$ for all $T\in 2^S\setminus\{\varnothing,S\}$ and all $j\in T$. Then, $\big(P(S,\mathbf{w}|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})\big)-\big(P(S\setminus\{j\},\mathbf{w}|_{S\setminus\{j\}},\succeq|_{S\setminus\{j\}})-P(S\setminus\{i,j\},\mathbf{w}|_{S\setminus\{i,j\}},\succeq|_{S\setminus\{i,j\}})\big)<\big(P(S,\mathbf{w}'|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}'|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})\big)-\big(P(S\setminus\{j\},\mathbf{w}'|_{S\setminus\{j\}},\succeq|_{S\setminus\{j\}})-P(S\setminus\{i,j\},\mathbf{w}'|_{S\setminus\{i,j\}},\succeq|_{S\setminus\{i,j\}})\big)$ for all $j\in S\setminus\{i\}$. But, it implies that

$$\begin{aligned}
&P(S,\mathbf{w}|_S,\succeq|_S)-P(S\setminus\{j\},\mathbf{w}|_{S\setminus\{j\}},\succeq|_{S\setminus\{j\}})\\
=&\big(P(S\setminus\{i\},\mathbf{w}|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})-P(S\setminus\{i,j\},\mathbf{w}|_{S\setminus\{i,j\}},\succeq|_{S\setminus\{i,j\}})\big)\\
&+\big[\big(P(S,\mathbf{w}|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})\big)\\
&-\big(P(S\setminus\{j\},\mathbf{w}|_{S\setminus\{j\}},\succeq|_{S\setminus\{j\}})-P(S\setminus\{i,j\},\mathbf{w}|_{S\setminus\{i,j\}},\succeq|_{S\setminus\{i,j\}})\big)\big]\\
=&\big(P(S\setminus\{i\},\mathbf{w}'|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})-P(S\setminus\{i,j\},\mathbf{w}'|_{S\setminus\{i,j\}},\succeq|_{S\setminus\{i,j\}})\big)\\
&+\big[\big(P(S,\mathbf{w}|_S,\succeq|_S)-P(S\setminus\{i\},\mathbf{w}|_{S\setminus\{i\}},\succeq|_{S\setminus\{i\}})\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \big(P(S \setminus \{j\}, \mathbf{w}|_{S\setminus\{j\}}, \succeq|_{S\setminus\{j\}}) - P(S \setminus \{i,j\}, \mathbf{w}|_{S\setminus\{i,j\}}, \succeq|_{S\setminus\{i,j\}})\big)\big] \\
&< \big(P(S \setminus \{i\}, \mathbf{w}'|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}}) - P(S \setminus \{i,j\}, \mathbf{w}'|_{S\setminus\{i,j\}}, \succeq|_{S\setminus\{i,j\}})\big) \\
&\quad + \big[\big(P(S, \mathbf{w}'|_S, \succeq|_S) - P(S \setminus \{i\}, \mathbf{w}'|_{S\setminus\{i\}}, \succeq|_{S\setminus\{i\}})\big) \\
&\quad - \big(P(S \setminus \{j\}, \mathbf{w}'|_{S\setminus\{j\}}, \succeq|_{S\setminus\{j\}}) - P(S \setminus \{i,j\}, \mathbf{w}'|_{S\setminus\{i,j\}}, \succeq|_{S\setminus\{i,j\}})\big)\big] \\
&= P(S, \mathbf{w}'|_S, \succeq|_S) - P(S \setminus \{j\}, \mathbf{w}'|_{S\setminus\{j\}}, \succeq|_{S\setminus\{j\}}),
\end{aligned}
$$

for all $j \in S \setminus \{i\}$. Consequently, $POSh_k(S, \mathbf{w}|_S, \succeq|_S) \prec_k POSh_k(S, \mathbf{w}'|_S, \succeq|_S)$ for all $k \in S$ (including i himself by premise), which is impossible. Therefore, the $POSh$ is not D-manipulable. $\square$

Proof of Proposition 7. Consider two economies $(N, \mathbf{w}, \succeq), (N, \mathbf{w}', \succeq) \in \mathcal{E}$ such that $\mathbf{w}_i > \mathbf{w}'_i$, $\mathbf{w}_i + \mathbf{w}_j = \mathbf{w}'_i + \mathbf{w}'_j$ for donor i and recipient j; $\mathbf{w}_k = \mathbf{w}'_k$ for each $k \in N \setminus \{i,j\}$. By considering the subeconomies without player j and without player i, Proposition 6 implies that $P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) - P(N \setminus \{i,j\}, \mathbf{w}|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}}) \geq P(N \setminus \{j\}, \mathbf{w}'|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) - P(N \setminus \{i,j\}, \mathbf{w}'|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})$.

Assume that the donor i is better off in $(N, \mathbf{w}', \succeq)$ than in $(N, \mathbf{w}, \succeq)$ under $POSh$, which means that $P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) < P(N, \mathbf{w}', \succeq) - P(N \setminus \{i\}, \mathbf{w}'|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Combining two inequalities together, we have $P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) + P(N \setminus \{i,j\}, \mathbf{w}|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}}) < P(N, \mathbf{w}', \succeq) - P(N \setminus \{i\}, \mathbf{w}'|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{j\}, \mathbf{w}'|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) + P(N \setminus \{i,j\}, \mathbf{w}'|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})$. But then,

$$
\begin{aligned}
&P(N, \mathbf{w}, \succeq) - P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) \\
=&P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{i,j\}, \mathbf{w}|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}}) \\
&+ \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) \\
&+ P(N \setminus \{i,j\}, \mathbf{w}|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})\big) \\
\leq&P(N \setminus \{i\}, \mathbf{w}'|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{i,j\}, \mathbf{w}'|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}}) \\
&+ \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) \\
&+ P(N \setminus \{i,j\}, \mathbf{w}|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})\big) \\
<&P(N \setminus \{i\}, \mathbf{w}'|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{i,j\}, \mathbf{w}'|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}}) \\
&+ \big(P(N, \mathbf{w}', \succeq) - P(N \setminus \{i\}, \mathbf{w}'|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{j\}, \mathbf{w}'|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})
\end{aligned}
$$

$$+ P(N \setminus \{i,j\}, \mathbf{w}'|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})\big)$$
$$= P(N, \mathbf{w}', \succeq) - P(N \setminus \{j\}, \mathbf{w}'\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}).$$

It means that recipient j must also be better off in $(N, \mathbf{w}', \succeq)$ under $POSh$. Therefore, the transfer paradox is not possible under $POSh$ □

Proof of Proposition 8. We first prove that the $POSh$ satisfies equation (1) in $\mathcal{E}^h$. Let P the potential function of the $POSh$. Then, for any $\mathbf{x}^N \in POSh(N, \mathbf{w}, \succeq)$ and any $(u_i)_{i\in N}$ that represents $\succeq$,

$$u_i(\mathbf{x}_i^N) = u_i\big(\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i\big)$$
$$= \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)u_i(\mathbf{w}_i),$$

where the last equality holds by homotheticity. A similar expression can be obtained for the allocations $\mathbf{x}_i^{N\setminus\{j\}}$, $\mathbf{x}_j^N$, and $\mathbf{x}_j^{N\setminus\{i\}}$. Therefore, (1) is equivalent to

$$\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)$$
$$- \big(P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) - P(N \setminus \{j,i\}, \mathbf{w}|_{N\setminus\{j,i\}}, \succeq|_{N\setminus\{j,i\}})\big)$$
$$= \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{j\}, \mathbf{w}|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})\big)$$
$$- \big(P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{i,j\}, \mathbf{w}|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})\big),$$

which trivially holds.

We now prove that if the solution F prescribes efficient allocations that satisfy equation (1), then it is necessarily the $POSh$. According to Definition 12, we need to find a potential function P such that

$$\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i \sim_i \mathbf{x}_i \tag{11}$$

for all $i \in N$ and all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$. We construct such a function P by induction on the number of players $|N|$. We define $P(\varnothing) = 0$. If $|N| = 1$, equation (1) implies that $\mathbf{x} = \mathbf{w}$ if $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$. Hence, we can define $P(N, \mathbf{w}, \succeq) = 1$ when $|N| = 1$ and the function P satisfies equation (11).

Consider now an economy $(N, \mathbf{w}, \succeq)$ with $|N| > 1$. Assume that the function P has been defined for any economy with a number of agents lower than $|N|$ and that it satisfies (11). Then, equation (1) is equivalent to:

$$\frac{u_i(\mathbf{x}_i^N)}{u_i(\mathbf{w}_i)} - \big(P(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}}) - P(N \setminus \{i,j\}, \mathbf{w}\,|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})\big)$$
$$= \frac{u_j(\mathbf{x}_j^N)}{u_j(\mathbf{w}_j)} - \big(P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) - P(N \setminus \{i,j\}, \mathbf{w}\,|_{N\setminus\{i,j\}}, \succeq|_{N\setminus\{i,j\}})\big).$$

Therefore,

$$\frac{u_i(\mathbf{x}_i^N)}{u_i(\mathbf{w}_i)} + P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) = \frac{u_j(\mathbf{x}_j^N)}{u_j(\mathbf{w}_j)} + P(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})$$

for any $i, j \in N$. Define

$$P(N, \mathbf{w}, \succeq) \equiv \frac{u_i(\mathbf{x}_i^N)}{u_i(\mathbf{w}_i)} + P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}),$$

where $i \in N$ (and the definition is correct because it does not depend on i). Then, equation (11) holds if

$$\frac{u_i(\mathbf{x}_i^N)}{u_i(\mathbf{w}_i)}\mathbf{w}_i \sim_i \mathbf{x}_i^N,$$

which is immediate for homothetic preferences. Hence, the function P is a potential, which concludes the proof. □

Proof of the relationship between the POSh and the PSh. We restrict attention to economies $(N, \mathbf{w}, \succeq)$ where each agent i's preference $\succeq_i$ is representable by a homothetic and quasi-linear utility function $u_i(\mathbf{x}) = w_i(\mathbf{x}\ |_{L\setminus\{m\}}) + x_m$. Define the TU game (N, v) by letting $v(S) \equiv \max_{\mathbf{z}\in Z(S,\mathbf{w}|_S)} \sum_{i\in S} u_i(\mathbf{z}_i)$ for each $S \in 2^N \setminus \{\varnothing\}$.

We are going to prove that $u_i(x) = PSh_i(N, v)$ for all $x \in POSh(N, \mathbf{w}, \succeq)$ and $i \in N$. Given that the PSh satisfies the properties of dummy player and dummy player out and the $POSh$ satisfies empty-agent and empty-agent out, we restrict attention here to economies where agents have positive endowments; the result is immediate for the empty agents. Moreover, Béal at al. (2018) show that the PSh can be characterized in terms of the "proportional potential," which is defined as a function P that satisfies that $\sum_{i\in N} \big(P(N, v) - P(N \setminus \{i\}, v)\big)v(\{i\}) = v(N)$. This definition of the potential is similar to that of the potential of the $POSh$. We proof the result by showing that the potential functions of the two environments coincide, that is, $P(N, \mathbf{w}, \succeq) = P(N, v)$. We prove it by induction on the number of agents $|N|$.

For $|N| = 1$, $(P(\{i\}, \mathbf{w}_i, \succeq_i) - P(\varnothing))\mathbf{w}_i \sim_i \mathbf{w}_i$ implies that $P(\{i\}, \mathbf{w}_i, \succeq_i) = 1$. For its induced TU game, $v(\{i\}) = u_i(\mathbf{w}_i) > 0$ (because $\mathbf{w}_i \neq \mathbf{0}$, $u_i(\mathbf{0}) = 0$ given that preferences are homothetic, and by strong monotonicity). Then, we have that $PSh_i(v) = v(\{i\}) = \big((P(\{i\}, v) - P(\varnothing)\big)v(\{i\})$ implies $P(\{i\}, v) = 1$.

For $|N| \geq 2$, we hypothesize that $P(S, \mathbf{w}\,|_S, \succeq|_S) = P(S, v)$ for all $S \subsetneq N$. Then, $P(N, \mathbf{w}, \succeq)$ satisfies that there exists $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ such that $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)\mathbf{w}_i \sim_i \mathbf{x}_i$, i.e., $\big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)u_i(\mathbf{w}_i) = u_i(\mathbf{x}_i)$ by homothetic preferences. The previous equality implies, by quasi-linearity, that $\sum_{i\in N} \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})\big)u_i(\mathbf{w}_i) = \sum_{i\in N} u_i(\mathbf{x}_i) = v(N)$. Using the induction argument we obtain $\sum_{i\in N} \big(P(N, \mathbf{w}, \succeq) - P(N \setminus \{i\}, v)\big)v(\{i\}) = v(N)$. Hence, the function $P(N, \mathbf{w}, \succeq)$ satisfies the defining equation of $P(N, v)$, which means that $P(N, \mathbf{w}, \succeq) = P(N, v)$. □

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